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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. It should be remembered that, at the present stage of mathematical research, ha[illegible]y paper is likely to be so completely original as to be independent o[illegible]earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with more familiar results

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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NOTICE.

1. In compliance with the wishes of members, the Managing Committee has decided to issue the Journal hereafter on a *new plan*—

(i) '**Original Articles**' will appear with continuous paging from number to number;

(ii) '**Notes and Questions**' will also appear with separate continuous paging—(the page numbers being printed in *italics* to avoid confusion).

2. Every number of the Journal will consist of *five* complete forms as usual.

3. The Progress Report of the Society will not form *part* of the Journal as hitherto; but will be issued as *extra* matter.

The Committee hopes that the *new plan* will meet with general approval.

P. V. SESHU AIYAR,
M. T. NARANIENGAR,
Hon. Joint Secretaries.

1*st Feb.* 1923.

5 - JUL

Time, Space, Matter and Mind.*

BY S. V. RAMAMURTHI, I. C. S.

I.

Before Einstein, the world was regarded as a three-dimensional world with an absolute framework and capable of analysis down to three fundamental entities': matter, space and time. Einstein asserts that the world is neither three-dimensional nor set in an absolute frame. He regards time as any direction of a four-dimensional space-time. Matter he conceives as a pecularity in this space-time. He in fact postulates a qualitative relationship between time, space and matter as against their absolute independence previously believed in.

I regard Einstein's analysis as correct as far as it goes, but I believe that analysis can go deeper and obtain simpler qualitative and quantitative relations between time, space, matter and, I would add, mind.

2. I consider that not only Einstein's Law of Gravitation, but also Newton's is more complex than it need be. At a time, when I had not heard of Einstein, Newton's Law of Gravitation which did not enable men with highly trained brains to determine the motion which n dead particles of matter could readily manipulate if left to themselves, seemed to me to involve an unnecessary complexity. It was in trying to recast Newton's Law that I have been led to my view of the relationship between time, space, matter and mind. From such a view, the general principle of relativity can be seen to be a truism as the conception of four-dimensional axes admits of being split up into simpler conceptions.

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* Since writing this paper, I have seen Professor Alexander's "*Space, Time, and Deity.*" His view agrees on several points with mine in Section II. This agreement is striking since I developed my view entirely independently of his. Our standpoints are different—his being philosophical and mine mathematical, and our views have also their divergence.

Professor A. N. Whitehead of the Imperial College of Science, London, has gone through my paper and discussed it with me and I am grateful to him for the suggestions he made to increase the clearness of presentation.

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II.

The hypothesis on which I base my theory of the relation of time, space and matter is as follows :—

" Matter, space and time are each aggregates of recognizable elements* of a common entity—call it mass—the ratios of the scales of size of the elements being very large."

This hypothesis involves

(*a*) the oneness of time, space and matter, and

(*b*) the manner of their difference.

Let me first show how I arrive at (*a*). The mathematical conception of time is expressed in Newton's First Law of Motion : " Every body continues in its state of rest or uniform motion in a straight line except in so far, as it is compelled by forces to change that state." If n bodies act under the action of no forces, then the distances moved in the same time are proportional to that time. These distances are proportional to themselves. We say body n moves from A_n to B_n as time changes from t_1 to t_2, where n may have any value from 1 to n. By this statement we mean that we perceive together the position of body 1 at A_1, body 2 at A_2, ..., body n at A_n, and the instant t_1. The perception of the instant t_1 means the perception of a stationary position in time, that is, a point in time. The perception of a point in time is not any the simpler then and is just the same physiologically as the perception of the position of a body at a point. We may therefore regard the statement, " we perceive together the positions of bodies 1 to n at A_1 to A_n and we perceive together their positions at B_1 to B_n" as simple a statement as " we perceive the bodies 1 to n at A_1 to A_n at one instant of time t_1 and we perceive their positions at B_1 to B_n at another instant of time t_2." †

The introduction of time into the expression of the identity of our perceptions is no more than the setting apart of one of the moving bodies as a standard of comparison. But a standard is not a separate entity from the many measures of length we have. We may therefore rewrite Newton's First Law as follows :—" Bodies moving under no influences in space move along straight lines and pass through proportional changes

* By an element I mean an aggregate which persists.

† The argument here is taken from my paper on " *Cosmic Synthesis*, " published in the Proceedings of the Benares Mathematical Society, Vol. I, 1919.

of space between positions perceived together." Time may be regarded as the change of space moved through by any one of the bodies. Change of time is thus a change of space. It may be argued that I have dealt not with time, but with a measure of time. But, if the time is always measured by means of space and only by means of space, then time is space. Time is standard space.

All our measures of space are made by means of matter. A volume of space is not a different entity from a volume of matter but is a standard volume ot matter. Space is thus standard matter.

Hence I state as the first part of my hypothesis that time, space and matter are all made of one entity : call it mass.

This mass we recognize by means of aggregates which persist. Thus a human body is a persisting aggregate. So is an atom or an electron. So is a centimeter or a second. By an element I mean an aggregate which persists.

The simplest relation which can be postulated between different aggregates of the same entity is the arithmetical relation. It has been recognized in philosophy that a large difference in quantity leads to a difference in quality. My hypothesis therefore lays down that a large quantitative difference between time, space and matter is the basis of their difference in quality.

To take an analogy, take the relation of pence, shillings and pounds They are all persisting aggregates of value. Starting from a penny, we can go to 2 pence, 3 pence,... 11 pence. When we reach 12 pence, the number of pence is too large for convenience and we change over to a shilling. Thus the smallest piece of shilling values is the largest piece of penny values. In my hypothesis, I conceive of time, space and matter analogously to pence, shillings and pounds, where, in place of the ratios 1 : 12 : 240, I have very large ratios.

An atom is the smallest piece we can split matter into. Its radius is of order 10^{-8} c.m. The radius of the largest piece of space is therefore of the same size.

The largest velocity we know of, *i. e.* the largest value of ds/dt is c, the velocity of light..

I have said that time and space are made of one entity. Hence we can obtain ds/dt by measuring both as space or both as time.

Taking both as space,

$$\text{the largest value of } \frac{ds}{dt} = \frac{\text{the largest value of } ds}{\text{the smallest value of } ds},$$

$$= \frac{\text{the smallest value of } dm}{\text{the smallest value of } ds},$$

$$= \frac{\text{a quantity of order } 10^{-8} \text{ cm.}}{\text{the smallest value of } ds},$$

where dm is an element of matter.

But the largest value of $ds/dt = c$, which is of order 10^{10} in C.G.S. units. Therefore the smallest value of ds is of order

$$10^{-10}\, 10^{-8}, \text{ i.e., } 10^{-18} \text{ cm.}$$

Again measuring ds/dt by taking ds, dt both as time,

$$\text{the largest value of } \frac{ds}{dt} = \frac{\text{the largest value of } dt}{\text{the smallest value of } dt}$$

$$= \frac{\text{the smallest value of } ds}{\text{the smallest value of } dt}$$

$$= \frac{\text{a quantity of order } 10^{-18} \text{ cm.}}{\text{the smallest value of } dt}.$$

Therefore the smallest value of dt is of order

$$10^{-18} \times 10^{-10} \text{ i.e., } 10^{-28} \text{ cm.}$$

Hence, if dm, ds, dt be the lengths of the smallest elements of matter, space and time, the orders of the ratios

$$dm : ds : dt$$

according to the second part of my hypothesis, are $10^{-8} : 10^{-18} : 10^{-28}$.

The human body is an element, *i.e.*, a persisting aggregate of matter of the scale of size 10^{2} cm. I regard it as the matter part of the human mind. It gives the latter its matter standpoint. Its scale of size determines human ideas of size. An atom is the smallest particle of matter relatively to the powers of perception attuned to the human body. If du represents the size of the human mind, the order of the ratios

$$du : dm : ds : dt,$$

where dm, ds, dt are the lengths of the smallest elements of matter, space and time, is

$$10^{2} : 10^{-8} : 10^{-18} : 10^{-28}$$

i.e.,

$$1 : 10^{-10} : 10^{-20} : 10^{-30}.$$

Relatively to the unit of human size, the size of the smallest particle of time is 10^{-30}. Time consists of the smallest particles of mass relatively to the powers of perception of the human mind. We can define time to be an aggregate of mass particles of the smallest size (10^{-28} cm.)

Space is an aggregate of mass particles of order of size 10^{-18} cm. matter of 10^{-8} cm. These figures hold in relation to the human mind.

We can explain how at crucial points of the growth of mass arithmetically we embark on new organic elements.

Imagine a room filled with grains of sand. Then let marbles be formed out of the sand all through the room. Then let cannon balls be formed from aggregates of marbles. Let the radii of a grain of sand, a marble and a cannon ball be in the ratios $1 : 10^{10} : 10^{20}$. Then the sand grains can be taken to represent time, the marbles, space and cannon balls matter.

A marble ball contains within it sand grains. Between the sand-grains as closely packed as possible, there is nothing, *i.e.*, there is such a small quantity of the elemental stuff of which sand-grains are formed as to be inappreciable. Zero is a relative quantity. Put a shilling in your palm and take it away. There is nothing in your hand. Repeat the process a sufficiently large number of times. The shilling gets worn a little. That little remains mostly in the hand. When we say nothing is left, we mean that less than the small quantity we regard as zero is left. But the thin stuff between sand-grains though negligible in volumes compared with those of sand grains may become appreciable in volumes very large compared with those of sand-grains. So too a layer of air has no colour. But a sufficiently thick layer of air is blue. Thus a sufficient aggregation of air leads to a new quality. If from a marble ball we remove all the grains of sand, we have left very thin stuff spread over a very large volume. If this thin stuff by such aggregation attains a mass finitely comparable to that of a sand-grain, we have the marble ball as the sum of the sand-grains plus an aggregate which is again an appreciable quantity. Applying the argument to space and time particles, space contains time plus a new aggregate which may be called *space* or "bare space." So too matter contains time, *space* (*i.e.* bare space) and a new aggregate which may be called *matter* or "bare matter." Thus matter is an aggregate of *matter*, *space* and time. Space is an aggregate of *space* and time.

The orthodox analysis of the universe into matter, space and time is really down to *matter-space*-time, *space*-time and time and is thus incomplete. To avoid confusion between matter as a simple entity (bare matter) and as the complex entity including space and time, I have italicised matter, when it represents a simple, *i.e.* a fundamental entity. So too for space.

If we regard time as yielding one dimension of the universe, then we can regard *matter* and *space* which stand on the same footing as time as also yielding each a dimension of the universe and hence we can regard matter *i.e.*, *matter-space*-time, as three dimensional. This implies that our material universe is three dimensional, because of its matter and that if matter be renewed from a material universe, it will no longer be three but two dimensional. I shall deal with the question of dimensions in the next section.

The ratios of the mass of a *matter* particle, a *space* particle and a time particle can, as shown, be viewed as finite and therefore may be taken to be of the order

$$1 : 1 : 1.$$

But their linear ratios of size are of the order $1 : 10^{-10} : 10^{-20}$.

Therefore their ratios of linear densities are of the order

$$1 : 10^{10} : 10^{20}.$$

Matter is thus froth in an ocean of *space* and *space* is froth in an ocean of time.

It is generally admitted that in the universe there is an element which is not time, space or matter. That entity is variously called life or mind or consciousness. I choose the term mind to represent that entity and take *mind* (*i.e.* bare mind) as the next term in the series of fundamental entities, time, *space, matter*. I then re-state my hypothesis by adding mind before matter, space and time. The human mind is an element of mind. To its size, the smallest particles of matter, space and time which it cognizes bear ratios of order

$$10^{2} : 10^{-8} : 10^{-18} : 10^{-28}$$

i.e.
$$1 : 10^{-10} : 10^{-20} : 10^{-30}.$$

I assume in my revised hypothesis (*i.e.* when including mind) that the ratio of elements of mind and matter is very large even if it is not of the order $1 : 10^{-10}$ which holds for the human mind.

III.

1. Take an aggregate of time particles. They are of order of size 10^{-28} cm. These small particles can be treated as equal and indistinguishable from each other. We can regard the aggregate as one dimensional for, if A be a particle and B a neighbouring particle, the line joining AB is indistinguishable from the line joining A to any other neighbouring point. From joining A to a neighbouring point, which we do to form a line, we get only one line, not more, as they are indistinguishable among themselves. To make a surface there, we must have at least two distinguishable lines. Having got an element of a line, the further evolution of the line can be traced as is at present done. Thus time may be conceived as a line winding through the aggregate of time mass.

Again, take space which is an aggregate of two kinds of particles—the time particle and the space particle (like the sand-grain and the marble). They are small, of order of size 10^{-28} and 10^{-18} cm. We can treat the time particles as all equal and indistinguishable from each other and we can treat the space particles as all equal and indistinguishable from each other. At each point we have two distinguishable line-elements and therefore an element of a surface which is the aggregate of two neighbouring line-elements. Having obtained an element of a surface, we can trace the evolutions of the surface from the element as we do at present. Thus space may be regarded as a surface winding through space mass. A surface is what we call a two-dimensional continuum.

Similarly matter may be regarded as an aggregate of three distinguishable lines, *i.e.*, as yielding a volume and hence as a three-dimensional mass.

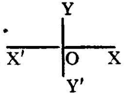
Take now matter mass in which at each point we have three independent lines. After plotting the lines, we can conceive the particles to be made of the same size or of two sizes. Thus we get time mass and space mass as also occupying a three dimensional frame like matter mass.

If mind be accepted as the next term in the series time, space, matter, we can arrive at mind as four-dimensional in the same way as we arrive at matter, space and time as three, two-and one-dimensional.

Thus the number of dimensions of an aggregate of mass is the number of entities time, *space*, *matter* and *mind* cognized in it. A fourth

dimensional universe is on this view a universe of mind, matter, space and time.

2. If in an aggregate of mass, we have time particles along an element of line XOX′ and space particles along a line YOY′ cutting the former at O,



then the change OX involves no change along OY, as time and space are of different classes and change in one is therefore independent of change in another.

We may define a right angle as the angle between OX and OY where a change in OX does not involve a change in OY.

If then OX, OY are lines of time and of space, then $\angle XOY =$ 1 right angle.

If OZ be a line of matter particles, OX, OY, OZ are similarly seen to be perpendicular to each other.

If in addition to this we make the time, *space* and *matter* particles equal, the three directions OX, OY, OZ have the same status and the aggregate is Euclidean. The difference between Euclidean and non-Euclidean aggregates depends on the units of measurements along perpendicular axes.

Hence an aggregate of *matter*, *space* and time particles yields a three axial non-Euclidean frame, if the particles are of unequal size and yields a three axial Euclidean frame, if they are of equal size.

Thus a three-dimensional aggregate of time particles yields an Euclidean three-dimensional frame. Any aggregate of mass can be reduced to an Euclidean continuum by making the particles equal after its orientation through differences of particles is first fixed.

IV.

1. A single law of conservation of mass yields the laws of conservation of energy, momentum and matter.

Energy varies as force and distance moved through

$$\therefore \text{ Energy} \propto \Sigma m.\, ds.\, \delta s,$$

where $\delta s = d\,(ds)$.

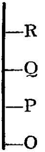
[*ds* is very small compared with a particle of matter.]

δs **is very small compared with** $ds.$; $m.ds.\delta s$ **is thus a three-dimensional mass ;**

$$\text{momentum} \propto m.\, ds\ ;$$
$$\text{matter} \quad \text{is } m.$$

The laws of conservation of matter, momentum and energy mean that, when an aggregate of mass is transformed, its three-dimensional, two-dimensional and one-dimensional parts are unaltered.

2. Time, *space* and *matter* evolve as mass grows arithmetically, that is, linearly.



Take a vertical line to represent this growth. Up to a point P, we have the region of time ; from P to Q, we have the region of space ; from Q to R that of matter.

Hence, *space* (which is the excess over time in space) is measured by the height over P which gives the level of time. *Matter* is measured by the height over Q which gives the level of space.

We can obtain the well-known result $m' = m$ and also the Einsteinian equations for the path of a planet from the relation of matter, space and time particles as shown in my paper on "Mathematical Points." *

V.

From the arithmetical relation between time, space and matter postulated in my hypothesis, I shall now deduce a geometrical relation between them.

Reduce time, space and matter to Euclidean continua as referred to at the end of Section III. Let t be an elementary length of time, r of space and m of matter.

Then an element of space being two dimensional (an area) varies as r^2, r being a length of the area. Similarly an element of matter which is three-dimensional (a volume) varies as m^3.

* Published in the Journal of the Indian Mathematical Society, October 1921.

∴ An element of time

: An element of space

: An element of matter

= certain arithmetical ratios, by my hypothesis.

$\therefore \quad t : r^2 : m^3 =$ certain arithmetical ratios.

$$\therefore \quad t = k_1 r^2 = k_2 m^3,$$

by proper choice of units of time, space and matter.

By changing the origin of measurement for t, we get

$$A + Bt = k_1 r^2 = k_2 m^3,$$

where A, B, k, k_2 are constants.

Taking mind as the next term in the series time, space, matter, mind is time-*space-matter-mind*.

Let u represent an elemental length of mind. Then

$$A + Bt = k_1 r^2 = k_2 m^3 = k_3 u^4.$$

Take a rod of time and spread it into a circle like a folded fan unfolded. Then a radius of the circle represents a rod of space. If we spread the circle (or the rod of time) into a sphere, a radius represents a rod of matter.

VI.

Take the equation connecting time and space as $t = k_1 r^2$.

If x, y be two perpendicular lengths in an Euclidean space, $k_1 x^2$, $k_1 y^2$ are the corresponding values of time. These two perpendicular lines correspond to time and *space*, *i.e.*, to time and the addition of mass to it which is *space*.

Hence $k_1 x^2 + k_1 y^2$

= total time change through the two passages x, y

$= k_1 r^2$, where r is the corresponding hypotenuse.

$$\therefore \quad r^2 = x^2 + y^2.$$

This is the basis of Euclidean Geometry.

Again take the equation

$$A + Bt = k_1 r^2$$

Let $r^2 = r^2_1 + r_2^2$

where r_1^2 is space,

r_2^2 is time.

Equating time and space,

$$B\,t = k_1 r_2^2, \quad \ldots \quad (1)$$

and

$$A = k_1 r_1^2. \quad \ldots \quad (2)$$

(A is the addition to time, as space is addition to time).

Equation (2) refers to timeless space and gives the constancy of r^2 in a timeless space.

As regards (1), we can take a function T^2 in place of t through measuring time by means of a space rod T. Hence **(1)** yields

$$BT^2 = k_1 r_2^2$$

i.e.,

$$T \propto r_2$$

This gives Newton's First Law of Motion (where time is measured by a space rod).

Again take

$$A + B\,t = k_1 r^2$$

Do not split r^2 into its time and space components.

Change t to T^2 by measuring time by a space rod (as we do in our mechanics).

$$A + BT^2 = k_1 r^2$$

$$\therefore \; k_1 r^2 - BT^2 = A \quad \ldots \quad (3)$$

If r and T be the smallest lengths of space and time,

$$\frac{r}{T} = c \text{ (= the velocity of light)}$$

$$\therefore \; r^2 - c^2\,T^2 = 0, \text{ in that case.}$$

Hence (3) should reduce to this, when A has the value zero.

Hence (3) can be written

$$r^2 - c^2T^2 = D, \text{ a constant,}$$

(where r, T are the smallest lengths of space and time).

Substituting $x^2 + y^2 + z^2$ for r^2 under the basic relation of Euclidean Geometry, we have $x^2 + y^2 + z^2 - c^2T^2 = D$.

This is in Einstein's original relation between space and time. It is on this relation that Einstein builds up the structure of a four-dimensional space-time and therefrom obtains his Law of Gravitation.

Thus our fundamental equation

$$A + Bt = k_1 r^3$$

yields the geometries of Euclid and of Einstein. Both are essentially two-dimensional and deal with *space*-time.

Now take the second and third terms in the geometrical equations obtained:

$$k_1 r^3 = k_2 m^3$$

$$\therefore \quad m \propto r^2$$

$$\therefore \quad \frac{dm}{dr} \propto r^{-1/3}$$

$$\therefore \quad \frac{d^2 m}{dr^2} \propto r^{-4/3}$$

$$\left(\propto - \frac{1}{m^2}\right)$$

Our space is measured by rods of matter and our time is measured by space.

$\therefore$ The result is equivalent to $\dfrac{d^2 s}{dt^2} \times \dfrac{-1}{s^2}$.

This gives the value of the acceleration as in Newton's Law of Gravitation.

Now take the equation (1st and 3rd terms of the geometrical equations)

$$t = k_2 m^3.$$

Let x, y, z be lengths along three perpendicular lines of a four-dimensional world. Then $k_2(x^3 + y + z^3)$

= sum of time values along three directions which represent time, *space* and *matter*.

These can be added up as time and the sum $= k_2 r^3$

$$\therefore \quad r^3 = x^3 + y^3 + z^3$$

or $\quad r^3 = x^3 + y^3, \quad$ if z is zero.

This yields the geometry of space-time in a world where rods of matter, space and time are measured by a perpendicular mind.

The equation

$$r^3 = x^3 + y^3$$

is the basis of four-dimensional geometry, as $r^2 = x^2 + y^2$ is the basis of three-dimensional geometry. A plane of straight lines requires a guiding length perpendicular to it, which yields both the directions and the means of measurement for the plane. For the two-directional plane of a three-dimensional mass, matter is the guiding and measuring rod. For the three-directional plane of a four-dimensional space, mind is the guiding and measuring rod.

A space-time in a four-dimensional world has the relation

$$r^3 = x^3 + y^3.$$

A study of this equation will lead us to four-dimensional logic, numbers, geometry and mechanics. I shall deal with them in the next section.

By measuring time, space and mind by means of matter, Einstein takes the equation $r^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$ and proceeds to his mechanics.

As before, in a n-dimensional universe, by taking further terms in the geometrical equations, we have

$$r^{n-1} = x_1^{n-1} + x_2^{n-1} + \dots\dots + x_{n-1}^{n-1}$$

and $\quad r^{n-1} = x^{n-1} + y^{n-1}$ in a space-time.

Thus it is possible to solve the equation

$$r^p = x^p + y^p.$$

where p is a positive integer, in terms of integral values of x, y and r, the numbers being $(p + 1)$ dimensional numbers.

Fermat's last theorem is untrue — provided that the integers are not numbers of matter.

(To be continued.)

On the Six Cross-Ratios of Four Points.

By R. Vydyanathaswamy.

1. If one of the cross-ratios of a range (ABCD) is λ, the others are

$$\frac{1}{\lambda},\ 1-\lambda,\ \frac{1}{1-\lambda},\ \frac{\lambda-1}{\lambda},\ \frac{\lambda}{\lambda-1}.$$

The homographic operations which derive the six cross-ratios from any one of them must clearly form a group of order six; since all these cross-ratios can be obtained as cross-ratios of the range AB′C′D′, where B′C′D′ is any permutation of BCD, this group is identical in structure with the permutation group of three symbols. As a matter of fact the triad $(0, 1, \infty)$ is transformed into itself by all these homographic operations and this is as it should be. For $0, 1, \infty$ are the possible cross-ratios of a range with two coincident points. Other sets of points which are transformed into themselves by these operations would be furnished by the co-variant sets of $(0, 1, \infty)$, namely, the Hessian points (or cyclic centres) $(-\omega, -\omega^2)$ and the cubi-covariant set $(-1, 2, \frac{1}{2})$ obtained by taking the harmonic conjugate of each of the points $0, 1, \infty$ either with respect to the other two, or with respect to the Hessian points. It will be noticed that $(-\omega, -\omega^2)$ are the possible cross-ratios of a self-harmonic range and $(-1, 2, \frac{1}{2})$ of a harmonic range.

2. *The pencil Γ_6 of the six cross-ratios.*

The previous paragraph shows that this pencil contains two perfect squares [corresponding to the triads $(0, 1, \infty)$ and $(-1, 2, \frac{1}{2})$] and one perfect cube (corresponding to the pair $-\omega, -\omega^2$). Hence by the 'Theory of Rational Transformations', (*J. I. M. S.*, April 1921) Γ_6 is reducible over a pencil Γ_3 of order 3.

Here Γ_3 is the pencil determined by the triads $(0, 1, \infty)$ and $(-1, 2, \frac{1}{2})$ and therefore contains two perfect cubes corresponding to the points $-\omega$ and $-\omega^2$. Hence we see that *the six cross-ratios are always roots of an equation of the form*

$$(x+1)^2 (x-1)^2 (2x-1)^2 + kx^2 (x-1)^2 = 0; \qquad \dots \quad (1)$$

or of the form

$$(x^2-x+1)^3 + k_1 x^2 (x-1)^2 = 0 \qquad \dots \quad (2)$$

or of the form

$$(x^2-x+1)^3 + k_2 (x+1)^2 (x-2)^2 (2x-1)^2 = 0 \qquad \dots \quad (3)$$

From (1) the rational function

$$R_1(\lambda) = \frac{(\lambda + 1)^2 \; (\lambda - 2)^2 \; (2\lambda - 1)^2}{\lambda^2 \; (\lambda - 1)^2}$$

is a symmetric function of the six cross-ratios and therefore an absolute invariant of the corresponding range. Otherwise expressed, $R_1(\lambda)$ is an automorphic function of the group of homographic transformations, which ransform any cross-ratio into any other.

Clearly $R_1(\lambda)$ is the only absolute invariant, *i.e.*, every other absolute invariant obtained from the pencil is merely a homographic function of $R_1(\lambda)$. In particular, we notice that the functions $R_2(\lambda)$, $R_3(\lambda)$ obtained from equations (2) and (3) are *real* homographic functions of $R_1(\lambda)$.

Ex. 1. Shew that any absolute invariant obtained from the pencil of cross-ratios is a reducible rational function of λ (in the sense of 'Rational Transformations' loc. cit). Hence shew that λ can be expressed by radicals as a function of the absolute invariant.

Considering $R_1(\lambda)$ for instance, we have

$$R_1(\lambda) = [P(\lambda)]^2$$

where

$$P(\lambda) = \frac{(\lambda + 1)(\lambda - 2)(2\lambda - 1)}{\lambda(\lambda - 2)}$$

$$= \frac{(\lambda + \omega)^3 + (\lambda + \omega^2)^3}{(\lambda + \omega)^3 - (\lambda + \omega^2)^3} \; 3(\omega - \omega^2).$$

Thus

$$\left(\frac{\lambda + \omega^2}{\lambda + \omega}\right)^3 = \frac{[R_1(\lambda)]^{\frac{1}{2}} - 3(\omega - \omega^2)}{[R_1(\lambda)]^{\frac{1}{2}} + 3(\omega - \omega^2)}.$$

From this λ may be expressed explicitly in terms of $R_1(\lambda)$.

Ex. 2. Theorem. If the absolute invariant $R_1(\lambda)$ of a range has a real value, then either, one and therefore all the cross-ratios are real, or one of the cross-ratios has unit modulus.

For, in this case $\lambda, \; \frac{1}{\lambda}, \; 1 - \lambda, \; \frac{1}{1-\lambda}, \; \frac{\lambda}{\lambda - 1}, \; \frac{\lambda - 1}{\lambda}$ are the roots of a real equation. Hence the conjugate $\bar{\lambda}$ of λ occurs elsewhere in the set. If $\bar{\lambda} = \frac{1}{1-\lambda}$ or $\frac{\lambda - 1}{\lambda}$, then clearly λ is real. If $\bar{\lambda} = \frac{1}{\lambda}$, then λ has unit modulus. If $\bar{\lambda} = 1 - \lambda$, then the conjugate of $\frac{\lambda}{\lambda - 1} = \frac{\lambda - 1}{\lambda}$; thus $\frac{\lambda}{\lambda - 1}$ has unit modulus. If $\bar{\lambda} = \frac{\lambda}{\lambda - 1}$, then $1 - \lambda$ has unit modulus.

The reason why the special function $R_1(\lambda)$ [including any real function of $R_1(\lambda)$] has been chosen in this theorem is that $R_1(\lambda)$ takes a real value for the special members of the pencil Γ_6 *viz.*, the perfect squares and the perfect cube. If it is desired to have a general statement of the theorem, we must express ourselves thus: If $R(\lambda)$ is any absolute invariant which takes the values $(a, b, c,)$ for the special members of Γ_6 and d is any number such that the cross-ratio $(a\ b\ c\ d)$ is real, then when $R(\lambda) = d$, either all the cross-ratios are real or one of them has unit modulus.

Ex. (3). If λ is a cross-ratio of the roots of the equation

$$a_0 z^4 + 4a_1 z^3 + 6a_2 z^2 + 4a_3 z + a_4 = 0,$$

show that $\frac{j^2}{i^3}$ is a numerical multiple of $\frac{(\lambda+1)^2(\lambda-2)^2(2\lambda-1)^2}{(\lambda^2-\lambda+1)^3}$ where i and j have the usual meanings.

3. *The Argand Plane.*

Theorem. *If λ be the cross-ratio of four points in the Argand plane, then the cross-ratio of their inverses in any circle is the conjugate of λ.*

Clearly this is true for reflection in the real axis. Hence by homographic transformation it is true for inversion in any circle. As particular cases of this theorem, we have:

(i) The cross-ratio (ABCD) is real, if and only if the four points are concyclic.

(ii) The cross-ratio has unit modulus if and only if there exists a circle through A, C which inverts B, D, into each other.

Hence if A, B, C are known and the absolute invariant of (ABCD) is real the locus of D consists of four circles, namely, the circle ABC and the three circles which pass through a vertex of ABC and invert the other two vertices into each other. These latter may be called the inversion circles of ABC; clearly they have two real common points D_1, D_2 [where $(ABCD_1) = -\omega$, $(ABCD_2) = -\omega^2$] and are inclined to one another at angles of 120°; (this follows, because successive inversion in two of these circles is a homographic operation of period 3). The points D_1, D_2 thus obtained are the Hessian points of ABC. They are also inverse points with respect to the circle ABC, because $-\omega$ is the conjugate of $-\omega^2$. Thus the inversion circles are all orthogonal to the circle ABC. Also D_1, D_2 are the points from which ABC can be inverted into an equilateral triangle; for the inverted triangle has one Hessian point at ∞ and must therefore be equilateral.

These results enable us to construct on the Argand plane that special variety of the pencil of cubics which contains two perfect cubes. Such a pencil is determined by any one of its members, ABC; and any other member PQR may be constructed from P in the same way as ABC is constructed from A. Thus Q, R, may be obtained from P by pairs of inversions in three circles passing through D_1, D_2 and making angles of 120° with each other. We notice also that the circle PQR belongs to the coaxal system which has D_1, D_2 for limiting points.

Let now Ω_1, Ω_2, Ω_3 be the inversion circles of ABC. Consider the effect on the triads of this pencil of the homographic operation composed of successive inversions in the circle ABC and one of the circles Ω_1, Ω_2, Ω_3. Since the inverting circles are orthogonal, the homography is an involution. Obviously this involution transforms into itself the triad ABC and the cubi-covariant triad composed of the remaining intersections of the circle ABC with Ω_1, Ω_2, Ω_3. Clearly no other triad of the pencil is self-corresponding. We may enunciate then the following theorem :—

The only involutoric correspondences between triads of a pencil of this type, of such a character that the self-corresponding triads are related as the cubi-covariant triads of each other, are those which are produced by successive inversion in a circle of the coaxal system passing through D_1, D_2 and in a circle of the orthogonal coaxal system.

We may apply these results to the study of the representation of Γ the pencil of the six cross-ratios in the Argand plane. It was seen that Γ_6 is reducible over a pencil Γ_3 which contains two perfect cubes and which is therefore of the type we have studied, the points D_1, D_2 being replaced by the points $-\omega$, $-\omega^2$. Further the two triads of Γ_3 which together constitute a member of Γ_6 are clearly connected by an involutoric correspondence, the self-corresponding triads of which are the square roots of the perfect squares of Γ_6, namely the triads $(0, 1, \infty)$ and $(-1, 2, \frac{1}{2})$. Since these are cubi-covariant triads of each other, the correspondence is of the type covered by the last theorem. Hence

The six points $\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda-1}{\lambda}, \frac{\lambda}{\lambda-1}$ *can be separated into two triads* $\left(\lambda, \frac{1}{1-\lambda}, \frac{\lambda-1}{\lambda}\right)$, $\left(\frac{1}{\lambda}, 1-\lambda, \frac{\lambda}{\lambda-1}\right)$ *the circumcircles of which determine the coaxal system whose limiting points are* $-\omega, -\omega^2$. *Further these circumcircles are reflections of one another in the real axis; and either of these triads can be made to coincide with the other by a reflection in the real axis followed by inversion in* any one *of the inversion circles of the triad* $(0, 1, \infty)$ *namely, the unit circles whose centres are* 0, 1, *and the line perpendicular to the real axis at the point* $\frac{1}{2}$.

Operational Methods in the Theory of Heat Conduction

By T. Surya Narayana.

1. The recent work of Dr. Bromwich * has shown that the operational method of Heaviside, who introduced it primarily in connection with electrical theory and also employed it in his discussion of the age of the Earth, is specially adapted for handling problems in the Theory of Heat Conduction. Various problems are found in Heaviside's *Electro-magnetic Theory* (Vol. 2), where he discusses the effects of steady and impulsive sources of electric charge and electro-motive force in linear conductors. It is proposed here to make a similar application of his symbolical method to determine the distribution of temperature in a solid due to an initial (instantaneous) source of heat. We shall confine ourselves to linear flow, and take for illustration some problems already solved otherwise.

2. Let K and c be the conductivity and heat capacity (per unit *volume*) of the substance, and put $k = K/c$. The temperature V satisfies the differential equation

$$k\frac{\partial^2 V}{\partial x^2} = \frac{\partial V}{\partial t}. \qquad \ldots \qquad \ldots \qquad \ldots \quad (1)$$

If we denote $\partial/\partial t$ by p, and regard it as subject to the ordinary laws of Algebra, and write $p = kq^2$, the above equation becomes

$$\frac{\partial^2 V}{\partial x^2} = q^2 V. \qquad \ldots \qquad \ldots \qquad \ldots \quad (2)$$

3. *Some Symbolical Formulæ.*

If P is a constant, and a solution in terms of normal functions exists, then

$$\frac{F(p)}{\Delta(p)}P = P\left\{\frac{F(o)}{\Delta(o)} + \sum\frac{F(\alpha)}{\Delta'(\alpha)}\frac{e^{\alpha t}}{\alpha}\right\}, \qquad \ldots \qquad \ldots \quad (3)$$

* *Phil. Mag.* (Ser. 6), 37, p. 407, 1919; and *Proc. Camb. Phil. Soc.*, 20, p. 411, 1921.

where the accent denotes differentiation with respect to p, α is a root of the equation $\Delta(p) = 0$, and the summation extends over all the roots.

This is Heaviside's Expansion Formula; Dr. Bromwich has given in his paper in the *Phil. Mag.* (cited before) an elementary proof applicable when $F(p)$ and $\Delta(p)$ are simple polynomials, and $\Delta(p)$ is of a higher degree than $F(p)$. Bromwich's proof suggests a slight extension of (3), *viz.*,

$$\frac{F(p)}{\Delta(p)}\phi(t) = \sum\left\{\frac{F(\alpha)}{\Delta'(\alpha)} e^{\alpha t}\int_0^t e^{-\alpha t}\phi(t)\,dt\right\}, \qquad \dots \quad (4)$$

with the restrictions indicated on the character of $F(p)$, and $\Delta(p)$.

For, we have

$$\frac{F(p)}{\Delta(p)} = \sum\frac{A_\alpha}{p-\alpha}, \text{ where } A_\alpha = \frac{F(\alpha)}{\Delta'(\alpha)};$$

and, if $z = \frac{1}{p-\alpha}\phi(t)$, then z is that solution of the equation

$$\frac{dz}{dt} - \alpha z = \phi(t)$$

which vanishes with t. Hence

$$z = e^{\alpha t}\int_0^t e^{-\alpha t}\phi(t)\,dt,$$

and

$$\frac{F(p)}{\Delta(p)}\phi(t) = \sum\left\{A_\alpha e^{\alpha t}\int_0^t e^{-\alpha t}\phi(t)\,dt\right\}.$$

The formulæ (3) and (4) can also be used when $F(p)$ and $\Delta(p)$ are functions of a more general type; but formal proofs are difficult.

We shall have occasion to use in the course of our work one other symbolical result.*

$$qe^{-qz}P = \frac{P}{\sqrt{\pi k t}}e^{-z^2/4kt} \qquad \dots \quad (5)$$

* Heaviside, *l. c.* p. 66. For a proof of this, and kindred formulae by translation into complex integrals, see Bromwich, *Proc. Camb. Phil. Soc*, 20, p. 425.

4. *Rod of length l; source at $x = \xi$; ends at zero temperture.*

Let V_1 be the temperature in the rod from $x = 0$ to $x = \xi$ at time t, and V_2 that from $x = \xi$ to $x = l$. We assume

$$V_1 = A \sinh qx, \qquad (0 \leqslant x \leqslant \xi)$$

$$V_2 = B \sinh q(l - x); \qquad (\xi \leqslant x \leqslant l)$$

for they are both solutions of (2), and vanish at $x = 0$ and $x = l$ respectively; A and B are independent of x, and have to be determined from the conditions at the source.

If we had a *steady* source of strength Q at $x = \xi$, the temperature should be continuous at $x = \xi$; and the rate of gain of heat by an element at $x = \xi$ from the flow must be equal to that from the impressed source. Hence V_1 and V_2 would have to satisfy at $x = \xi$ the conditions

$$\left.\begin{aligned} V_1 &= V_2 \\ \text{and} \quad k\frac{\partial V_1}{\partial x} - k\frac{\partial V_2}{\partial x} &= c.\,Q, \\ \text{or} \quad \frac{\partial V_1}{\partial x} - \frac{\partial V_2}{\partial x} &= \frac{1}{k}\,Q. \end{aligned}\right\} \quad \ldots \quad (6)$$

But the source at $x = \xi$ is *instantaneous*. Now if Q be any function of the time, pQ is its rate of increase. Consider a function Q which is zero before $t = 0$, and constant for $t > 0$. Then pQ is zero for $t > 0$; at $t = 0$ it is infinite but its time total is Q. Thus pQ means a function of t which is wholly concentrated at the moment $t = 0$, and whose total amount is Q; that is, it represents an instantaneous source of strength Q.

Therefore replacing Q by pQ in (6) we have the conditions to be satisfied at $x = \xi$:

$$\left.\begin{aligned} V_1 &= V_2 \\ \frac{\partial V_1}{\partial x} - \frac{\partial V_2}{\partial x} &= \frac{1}{k}\,pQ = q^2 Q \end{aligned}\right\} \quad \ldots \quad \ldots \quad (7)$$

Substituting in these the values of V_1 and V_2, we find

$$A = \frac{\sinh q\,(l - \xi)}{\sinh ql}\,qQ, \quad B = \frac{\sinh q\xi}{\sinh ql}\,qQ.$$

Thus

$$V_1 = \frac{\sinh qx \ \sinh q\,(l-\xi)}{\sinh ql}\, qQ, \quad \dots \quad \dots \quad \dots \quad (8)$$

$$V_2 = \frac{\sinh q\xi \ \sinh q\,(l-x)}{\sinh ql}\, qQ. \quad \dots \quad \dots \quad \dots \quad (9)$$

It will be observed that, if we interchange x and ξ in V_1, we obtain V_2.

To interpret these solutions, we employ the formula (3).

Take V_1. $F(p) = \sinh qx \ \sinh q\,(l-\xi)\, q$; $\Delta(p) = \sinh ql$.

The roots of $\Delta(p) = 0$ are given by

$$ql = n\pi i, \quad p = kq^2 = -kn^2\pi^2/l^2.$$

Also $\dfrac{F(0)}{\Delta(0)} = 0$;

and
$$\frac{F(p)}{\Delta'(p)} = \frac{\sinh qx \sinh q\,(l-\xi)\, q}{\dfrac{d}{dq}(\sinh ql).\ \dfrac{dq}{dp}}$$

$$= \frac{\sinh qx \ \sinh q\,(l-\xi).\ 2\,kq^2}{l \cosh ql},$$

so that

$$\frac{1}{a}\,\frac{F(a)}{\Delta'(a)} = -\frac{2}{l}\;\frac{\sin \dfrac{n\pi x}{l}\ \sin\left(n\pi - \dfrac{n\pi\xi}{l}\right)}{\cos n\pi}$$

$$= \frac{2}{l}\sin\frac{n\pi x}{l}\ \sin\frac{n\pi\xi}{l}.$$

Therefore

$$V = \frac{2Q}{l}\sum_{n=1}^{\infty} \sin\frac{n\pi\xi}{l}\ \sin\frac{n\pi x}{l}\ e^{-kn^2\pi^2 t/l^2}; \quad \dots \quad (10)$$

the summation being over positive values of n, since negative values do not furnish any fresh values of p.

The suffix for V has been dropped since it is symmetrical in x and ξ, and hence represents the temperature over the whole rod.

We can allow for an arbitrary initial temperature $f(x)$ in the rod by taking $Q = f(\xi)\, d\xi$, and intergrating over the length of the rod.

5. *Ends at $\phi_1(t)$ and $\phi_2(t)$. Source at $x = \xi$.*

If $x=0$ and $x=l$ are maintained at temperatures $\phi_1(t)$, and $\phi_2(t)$ respectively, we assume

$$V_1 = \phi_1(t) \cosh qx + C \sinh qx, \qquad (0 \leqslant x \leqslant \xi)$$

$$V_2 = \phi_2(t) \cosh q(l - x) + D \sinh q(l - x), \qquad (\xi \leqslant x \leqslant l).$$

V_1 and V_2 have to satisfy the conditions

$$V_1 = V_2 \text{ at } x = \xi,$$

$$\frac{\partial V_1}{\partial x} - \frac{\partial V_2}{\partial x} = q^2 Q \text{ at } x = \xi.$$

These give C and D; and we have

$$V_1 = \frac{\sinh q(l - x)}{\sinh ql}\phi_1(t) + \frac{\sinh qx}{\sinh ql}\phi_2(t) + \frac{\sinh qx \sinh q(l - \xi)}{\sinh ql}\, q\, Q, \quad \dots (11)$$

$$V_2 = \frac{\sinh q(l - x)}{\sinh ql}\phi_1(t) + \frac{\sinh qx}{\sinh ql}\phi_2(t) + \frac{\sinh q\xi \sinh q(l - x)}{\sinh ql}\, q\, Q. \quad \dots (12)$$

The first two terms in V_1 can be evaluated in terms of normal functions by means of the formula (4); and the last term is, as we have seen, represented by (10). On doing the algebra, we find

$$V = \sum_{n=1}^{\infty} \frac{2kn\pi}{l^2} \sin\frac{n\pi x}{l}\, e^{-kn^2\pi^2 t/l^2} \int_0^t e^{kn^2\pi^2 t/l^2} \times$$

$$\{\phi_1(t) - (-1)^n \phi_2(t)\}\, dt + \sum_{n=1}^{\infty} \frac{2Q}{l} \sin\frac{n\pi x}{l} \sin\frac{n\pi\xi}{l}\, e^{-kn^2\pi^2 t_0/l^2}$$

$$= \frac{2}{l} \sum_{1}^{\infty} e^{-kn^2\pi^2 t/l^2} \sin \frac{n\pi x}{l} \Big[Q \sin \frac{n\pi \xi}{l}$$

$$+ \frac{kn\pi}{l} \int_0^t e^{kn^2\pi^2 t/l^2} \{ \phi_1(t) - (-1)^n \phi_2(t) \} \, dt \Big] . \qquad \ldots (13)$$

It is clear that the expansion (13) represents also V_2, and therefore the temperature in the whole rod.

6. *Source at $x = \xi$, the end $x = l$ at zero, radiation at $x = 0$ into a medium at zero.*

Here we take

$$V_1 = A \cosh qx + B \sinh qx, \qquad (o \leqslant x \leqslant \xi)$$
$$V_2 = C \sinh q(l - x), \qquad (\xi \leqslant x \leqslant l).$$

If H is the emissivity of the substance, and $h = H/K$, the condition to be satisfied at the end $x = 0$ is

$$\frac{\partial V_1}{\partial x} = h V_1.$$

Also, as before, we should have at the source

$$V_1 = V_2 \text{ and } \frac{\partial V_1}{\partial x} = \frac{\partial V_2}{\partial x} = q^2 Q.$$

Finding A, B, and C from these three conditions, we obtain

$$V_2 = \frac{q \cosh q\xi + h \sinh q\xi}{q \cosh ql + h \sinh ql} \sinh q(l - x) \, qQ \qquad \ldots (14)$$

To apply (3), we can take

$$F(p) = (q \cosh q\xi + h \sinh q\xi) \, q \sinh q(l - x)$$
$$\Delta(p) = q \cosh ql + h \sinh ql.$$

So that

$$\frac{F(o)}{\Delta(o)} = 0; \text{ and } \frac{F(p)}{\Delta'(p)} = \frac{(q \cosh q\xi + h \sinh q\xi) \, q \sinh q(l - x)}{\frac{d}{dq}(q \cosh ql + h \sinh ql) \frac{dq}{dp}}$$

$$= \frac{2kq^2 (q \cosh q\xi + h \sinh q\xi) \sinh q(l - x)}{(1 + hl) \cosh ql + ql \sinh ql}.$$

Notes and Questions.

Note on a Formula in Solid Geometry.

To find the angle between the lines whose direction cosines are given by the equations

$$ul + vm + wn = 0, \qquad \ldots \quad (1)$$

and

$$al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0. \qquad \ldots \quad (2)$$

The following variation of the method given in Bell's *Three Dimensions. Art. 60*, and Frost's *Solid Geometry, Art. 26* may be interesting.

The result of eliminating $l : m : n$, between (1), (2) and

$$u'l + v'm + w'n = 0 \qquad \ldots \quad (3)$$

is easily seen to be

$$\begin{vmatrix} a & h & g & u & u' \\ h & b & f & v & v' \\ g & f & c & w & w' \\ u & v & w & 0 & 0 \\ u' & v' & w' & 0 & 0 \end{vmatrix} = 0. \qquad \ldots \quad (4)$$

[*Vide* Scott's *Determinants*, p. 145].

Replacing (3) by the identical relation

$$(m)\, l - (l)\, m = 0 \qquad \ldots \quad (5)$$

the result of eliminating n between (1) and (2) is deduced from (4) in the form

$$\begin{vmatrix} a & h & g & u & m \\ h & b & f & v & -l \\ g & f & c & w & 0 \\ u & v & w & 0 & 0 \\ m & -l & 0 & 0 & 0 \end{vmatrix} = 0,$$

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which is equivalent to

$$Bl - 2Hlm + Am = 0. \qquad \ldots \quad (6)$$

where A, B, C, etc., are the co-factors of a, b, c, etc., in the determinant

$$\Delta \equiv \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix} \qquad \ldots \quad (7)$$

From (5) and two other similar equations, if (l_1, m_1, n_1), (l_2, m_2, n_2) be the direction cosines of the two lines, we have

$$\frac{l_1 l_2}{A} = \frac{m_1 m_2}{B} = \frac{n_1 n_2}{C} = \frac{m_1 n_2 + m_2 n_1}{2F}$$

$$= \frac{n_1 l_2 + n_2 l_1}{2G} = \frac{l_1 m_2 + l_2 m_1}{2H}.$$

Hence if θ be the angle between the lines,

$$\tan^2 \theta = \frac{\Sigma (m_1 n_2 - m_2 n_1)^2}{(l_1 l_2 + m_1 m_2 + n_1 n_2)^2}$$

$$= \frac{4 \Sigma (F^2 - BC)}{(A + B + C)^2} \qquad \ldots \quad (8)$$

Again from (7), the principles of the reciprocal determinant gives $F^2 - BC = u^2 \Delta$, etc.

$$\text{Hence} \quad \tan \theta = \frac{2 \sqrt{\Delta (u^2 + v^2 + w^2)}}{A + B + C}. \qquad \ldots \quad (9)$$

F. H. V. GULASEKHARAM.

The Operator $(x\,D)^n$.

The object of this note is to prove that

$$(x\,D)^n = x^n D^n + \Sigma(n-1)\, x^{n-1} D^{n-1} + \Sigma(n-2)\,\Sigma(n-2)\, x^{n-2} D^{n-2} + \ldots$$
$$+ \Sigma(n-r)\,\Sigma(n-r) \ldots\ldots \Sigma(n-r).\ x^{n-r} D^{n-r} + \ldots + x\,D \quad (1)$$

the number of Σ's in the co-efficient of $x^{n-r} D^{n-r}$ being r, and these indicate multiple summation. (See page 55, Vol. XIV. *J. I. M. S.*).

The co-efficients are easily calculated by the help of the following table, which is self explanatory :—

n	Σn	$n\Sigma n$	$\Sigma n\Sigma n$	$n\Sigma n\Sigma n$	$\Sigma n\Sigma n\Sigma n$	$n\Sigma n\Sigma n\Sigma n$	$\Sigma n\Sigma n\Sigma n\Sigma n$
1	1	1	1	1	1	1	1
2	3	6	7	14	15	30	31
3	6	18	25	75	90	270	301
4	10	40	65	260	350	1400	1701
5	15	75	140	700	1050	5250	
6	21	126	266	1596	2646		
7	28	196	462	3234	5880		
8	36	288	750	6000	11880.		

e. g. $(x\,D)^4 = x^4 D^4 + 6\,x^3 D^3 + 7\,x^2 D^3 + x\,D.$

$(x\,D)^5 = x^5 D^5 + 10\,x^4 D^4 + 25\,x^3 D^3 + 15\,x^2 D^2 + x\,D.$

Now obviously from the way in which the summation takes place, as illustrated in the table, we have the general formula of reduction

$$\Sigma(n-r)\,\Sigma(n-r)\ldots\Sigma(n-r) = (n-r)\,\Sigma(n-r)\,\Sigma(n-r)\ldots\Sigma(n-r)$$
$$+ (n-r-1)\,\Sigma(n-r-1)\,\Sigma(n-r-1)\ldots\ldots\Sigma(n-r\,1)$$
$$+ \ldots\ldots + 2\,\Sigma 2\,\Sigma 2 \ldots\ldots \Sigma 2 + 1\,\Sigma 1\,\Sigma 1\,\Sigma 1 \ldots\ldots \Sigma 1, \quad (2)$$

the number of Σ's on the left is r, and in each of the terms on the right is $r-1$. This may also be written,

$$\Sigma(n-r)\,\Sigma(n-r)\ldots\ldots\Sigma(n-r) = (n-r)\,\Sigma(n-r)\,\Sigma(n-r)\ldots\ldots$$
$$+ \Sigma(n-r-1)\,\Sigma(n-r-1)\ldots\ldots\Sigma(n-r-1) \quad (3)$$

the number of Σ's on the left and in the second term on the right is r, and in the first term, it is $r-1$. Remembering (3), it is very easy to prove (1) by induction. For, from (1),

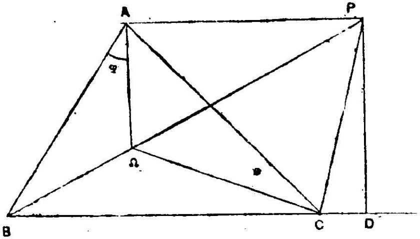
$$(x\,D)^{n+1} = x^{n+1} D^{n+1} + \{n + \Sigma(n-1)\}\, x^n D^n + \ldots\ldots\ldots\ldots$$
$$+ x^{n-r+1} D^{n-r+1}\ \{\Sigma(n-r)\,\Sigma(n-r)\ldots\ldots\Sigma(n-r)$$
$$+ (n-r+1)\,\Sigma(n-r+1)\,\Sigma(n-r+1)\ldots\Sigma(n-r+1)\} + \ldots + x\,D$$

there being r Σ's, in the first and $(r-1)$ in the second term in the coefficient of $x^{n-r+1} D^{n-r+1}$, whence (1) follows by induction.

C. KRISHNAMACHARI

A Geometrical Proof of the Property $\cot w = \cot A + \cot B + \cot C$.

Let $B\Omega$ produced cut the parallel through A to BC at P and let PD be perpendicular to BC produced.



Then, $\angle PCD = \angle A$ (from construction since APC Ω is cyclic).

$$\therefore \quad \cot A = \cot PCD = CD/PD.$$

$$\text{But} \quad \cot B + \cot C = BC/PD.$$

$$\therefore \quad \cot A + \cot B + \cot C = BD/PD = \cot w.$$

V. V. SATHYANARAYANA MURTHY.

Solutions.

Question 1122.

(V. RAMASWAMI AIYAR) :—If the joins of four concylic points A, B, C, D taken in opposite pairs intersect in P, Q, R, prove that the nine-points circle of PQR passes through the centroid of the points A, B, C, D.

Solution by S. Audinarayanan and B. Venkatescaran.

Let ABCD be the cyclic quadrilateral and PQR its diagonal Δ. The mid.-points M, N, X of AC, BD, QR respectively are collinear, G the centroid of ABCD is the mid.-point of MN. Let α be the middle point of OP, where O is the centre of the circle passing through A, B, C, D. Join OM, ON, αM, αN, αG.

Angles OMP and ONP are right angles.

$\therefore$ $\alpha M = \alpha N$, each being half of OP.

Now in Δ MαN, $\alpha M = \alpha N$ and G is the mid.-point of MN.

$\therefore$ αG is $\perp$ to MNX.

With respect to the Δ PQR, O is the ortho-centre.

The nine-point circle of Δ PQR, will be the circle on Xα as diameter, since X is the mid.-point of QR and α of OP. Hence G lies on the nine-point circle of Δ PQR, since $\angle XG\alpha$ has been proved to be a right-angle.

Question 1139.

(R. VYTHYNATHASWAMY) :—If $x_0, x_1, \ldots x_n$ are the homogeneous co-ordinates in n dimensions, find the complete curve of intersection of the following surfaces:—

$$\left.\begin{array}{ll} x_0x_2 - x_1^2 & = 0 \\ x_1x_3 - x_2^2 & = 0 \\ \ldots \quad \ldots \quad \ldots & \\ x_{n-2}\, x_n - x^2_{n-1} & = 0. \end{array}\right\}$$

Remarks by the Proposer.

Mr. Narasinga Rao's solution in the issue of October 1921 is incomplete, inasmuch as the complete curve of intersection should be of order 2^{n-1} whereas the twisted n-*ic* is only of order n.

To obtain the complete intersection, we notice that if x_1 is not zero then none of the co-ordinates of the common point vanishes identically, so that we can correctly say that the common point satisfies the equations $\frac{x_0}{x_1}=\frac{x_1}{x_2}=$ etc. giving the twisted n-*ic*. This also shews that all the common points which do not lie on the twisted n-*ic*, must satisfy $x_1=0$ and therefore $x_1=x_2=\ldots=x_{n-1}=0$. These last equations determine the line $x_0\,x_n$ —which is directly seen to be on all the quadrics. Thus the line $x_0\,x_n$ counted an appropriate number of times together with the twisted n-*ic* will constitute the complete curve of intersection.

The question suggests an analogy of the twisted n-*ic* curve with the ordinary twisted cubic. We know that the twisted cubic in three dimensions is the residual intersection of two quadrics having a common generating line.

The question shews the truth of a similar property of the twisted n-*ic* curve, *viz.* "*If $n-1$ quadrics in n dimensions have a common generating line L and if all of them touch at two points a, b on L, then the residual intersection of the quadrics is a twisted n-ic through a, b.*"

It will be an interesting, though probably, a very difficult exercise, to shew by direct reasoning, that the multiplicity of the line x_0x_n (regarded as a common curve of the quadrics), is exactly $2^{n-1}-n$.

Question 1157.

(K. J. SANJANA AND K. B. MADHAVA) :—In all integral right-angled triangles in which the sides are prime to each other, the sums and the differences of the sides are (1) either primes every one of which has 2 for a quadratic residue, or (2) the products of such primes and their powers.

Solution by A. A. Krishnaswami Aiyangar.

The sides of integral right-angled triangles in which the sides are prime to each other, are represented by the well-known expressions

$$x^2+y^2,\ x^2-y^2,\ \text{and}\ 2xy,$$

where $x > y$ and x and y are prime to each other and not both odd.

Now the sums and differences of the sides (not the hypotenuse) are given by the expression $(x^2 - y^2 \pm 2xy)$ whose determinant is 2, which is by a well-known theorem in the Theory of Numbers, a quadratic residue of all numbers represented by the quadratic form. (*Vide*: Art. 154, *Disquisitiones Arithmeticæ* of Ch. Fr. Gauss).

Question 1161.

(K. J. SANJANA) :—Prove that

$$\frac{n(n+1)\ldots(2n-1)}{n!} + \frac{n(n+1)\ldots\ldots 2n}{(n+1)!}\cdot\frac{1}{2}$$

$$+ \frac{n(n+1)\ldots(2n+1)}{(n+2)!}\cdot\frac{1}{2^2} + \ldots \textit{ad inf.} = 2^{2n-1}.$$

Solution by M. V. Ramakrishnan.

$$S = {}_{2n-1}C_n + {}_{2n}C_{n+1}\cdot\frac{1}{2} + {}_{2n+1}C_{n+2}\cdot\frac{1}{2^2} + \ldots\ldots\ldots$$

$$= \text{co-efficient of } x^n \text{ in}$$

$$(1+x)^{2n-1} + \frac{(1+x)^{2n}}{2x} + \frac{(1+x)^{2n+1}}{(2x)^2} + \ldots\ldots$$

i.e., $$\text{in } (1+x)^{2n-1}\left\{1 + \frac{1+x}{2x} + \left(\frac{1+x}{2x}\right)^2 + \ldots\ldots\right\}$$

i.e., $$\text{in } (1+x)^{2n-1}\left\{1 - \frac{1+x}{2x}\right\}^{-1} \text{ if } x > 1$$

i.e., $$\text{in } 2(1+x)^{2n-1}\left(1 - \frac{1}{x}\right)^{-1}.$$

Hence $S = 2\left({}_{2n-1}C_n + {}_{2n-1}C_{n+1} + \ldots\ldots + {}_{2n-1}C_{2n-1}\right) = 2^{2n-1}.$

Question 1163.

(V. RAMASWAMI AIYAR):—If a rectangular hyperbola passes through the in-centre of a triangle and the feet of the perpendiculars drawn therefrom to the sides, prove that it cuts the inscribed circle again at the point which is diametrically opposite the Feuerbach point.

Additional Solutions, (1) by F. H. V. Gulasekharam and (2) by R. Gopalaswami.

(1) The Feuerbach point of any triangle is the Euler point of the triangle formed by the points of contact of the incircle with the sides (*J. I. M. S.*, Vol. IX, p. 58). Hence it is sufficient to prove the following theorem :—

If a rectangular hyperbola passes through the vertices of a triangle and its circum-centre, it cuts the circum-circle again at the point which is diametrically opposite the Euler point of the triangle.

Since the *R. H.* passes through the circum-centre, the *R. H.* and the Euler line of the triangle are isogonal transformations of each other.

Also the circum-circle and the line at infinity are isogonal transformations of each other.

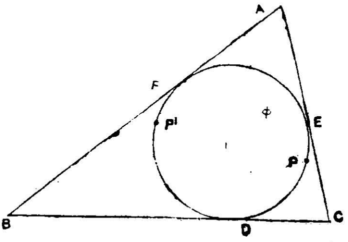
Hence if the *R. H.* cuts the circum-circle again at D, the isogonal conjugate of D is the point at infinity on the Euler line.

Therefore the Simson's line of D with respect to ABC is perpendicular to the Euler line, which latter is parallel to the Simson's line of the Euler point.

Hence the Simson's lines of D and of the Euler point cut at right angles.

Therefore D is the point on the circum-circle which is diametrically opposite to the Euler point. *Cf.* Gallatly : *Modern Geometry of the Triangle*, p. 59.

(2) The above theorem is allied to the following theorem and is easily deducible from it.



D, E and F are the points of contact of Δ ABC with the in-circle; p the Feuerbach point; p' its diametrically opposite point; ϕ the orthocentre of ΔDEF.

It can be readily proved by comparing angles that the diameters through D, E and F to the circum-circles of Δs DIϕ, EIϕ and FIϕ are concurrent on the circle DEF.

Further the point of concourse is p the Feuerbach point.

Since Fp passes through the circum-centre of FIϕ, the isogonal conjugate of Fp is at right angles to Iϕ.

Hence the isogonal conjugate of Fp' is parallel to Iϕ.

Similarly, the isogonal conjugates of Dp' and Ep' are parallel to Iϕ; that is, the isogonal conjugate of p' lies on Iϕ at infinity.

Thus p' lies on the isogonal transform of Iϕ with respect to ΔDEF.

I and ϕ transform into each other.

Consequently p' lies on the conic through D, E, F, I, and ϕ.

The rectangular hyperbola through D, E, F, and I therefore passes through p'.

Again, Iϕ passes through the centroid of ΔDEF.

Hence the isogonal transform passes through the Lemoine point of ΔDEF, which is the Gergonne point of ΔABC.

Question 1191.

(G. V. TELANG) :—ABC is a triangle inscribed in a circle. Three tangents are drawn to the circle so that the portions of them cut off by AB, AC are bisected at the points P, Q and R. Shew that the ortho-centre of the triangle PQR lies at the middle point of BC.

Additional Solution by F. H. V. Gulasekharam.

Take O the centre of the circle for origin, and the radius of the circle for unit length.

Let A, B, C be the *affixes* in the Argand Diagram of the nnmber 1 *(unity)*, $b \equiv \cos\alpha + i\sin\alpha$, and $c \equiv \cos\beta + i\sin\beta$ respectively.

Let P be a point $z \equiv \cos\lambda + i\sin\lambda$, such that the portion of the tangent at P intercepted by AB, AC is bisected at P.

Let P′ be the point z^2; then it is readily seen that AP′ is parallel to the tangent at P.

Hence the pencil A(BPCP′) is harmonic.

∴ The cross-ratio $(bc, zz^2) = -1$.

∴ $$\tfrac{1}{2}(b+c)(z+z^2) = bc + z^3 \qquad (1)$$

[*Vide* HARDY'S *Course of Pure Mathematics, 2nd Edition, p. 96.*]

Equation (1) gives three different values for z.

Let P, Q, R be the points corresponding to these three values z_1, z_2, z_3.

Let G be the centroid and H the ortho-centre of the triangle PQR.

Now G is the point $\tfrac{1}{3}(z_1 + z_2 + z_3)$, and

$$OG : GH = 1 : 2.$$

∴ H is the point $z_1 + z_2 + z_3$.

But from (1), $z_1 + z_2 + z_3 = \tfrac{1}{2}(b+c)$;

and $\tfrac{1}{2}(b+c)$ is the middle point of BC.

∴ H is at the middle point of BC.

NOTE :—Since $z_1 z_2 z_3 = -bc$,

$$\lambda_1 + \lambda_2 + \lambda_3 = (2n+1)\pi + \alpha + \beta.$$

Question 1195.

(K. J. SANJANA, M.A.) :—From the point P (p, q, r) perpendiculars PM, PN are drawn to the planes given by the equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0;$$

prove that the square of the length of MN is given by

$$\frac{A^{-1}(qG - rH)^2 + B^{-1}(rH - pF)^2 + C^{-1}(pF - qG)^2}{A + B + C + \frac{1}{4}(a + b + c)^2},$$

where A, B, C, F, G, H have their usual values.

Solution by F. H. V. Gulasekharam.

Let L be the foot of the perpendicular from P on the line of intersection of the given planes.

Then it is readily seen that the points P, M, L, N are concyclic and that

$$MN = PL \sin MLN. \quad \ldots \quad \ldots \quad (1)$$

Again since the given equation represents a pair of planes,

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

$$\therefore \quad BC = F^2 \,;\, CA = G^2 \,;\, AB = H^2. \quad \ldots \quad (2)$$

The direction cosines of the line of intersection of the planes are easily seen to be

$$\lambda A^{\frac{1}{2}}, \quad \lambda B^{\frac{1}{2}}, \quad \lambda C^{\frac{1}{2}},$$

where $\lambda^2 = \dfrac{1}{A + B + C}$ ... ... (3)

$\therefore \quad PL^2 = \lambda^2 \Sigma (qB^{\frac{1}{2}} - rC^{\frac{1}{2}})^2$ (*Vide* BELL'S *Three Dimensions*, *p.* 24)

$$= \lambda^2 \, \Sigma \{ A^{-1} (qG - rH)^2 \}, \text{ from } (2); \quad \ldots \quad (4)$$

also $\sin^2 MLN = \dfrac{A + B + C}{A + B + C + \frac{1}{4}(a + b + c)^2}$.

(*Vide* C. SMITH'S *Solid Geometry*, *p.* 36, *Ex.* 20.)

Hence from (1), (3), (4) and (5), the required result follows.

Additional Solutions by B. B. Bagi, G. S. Mahajani, V. A. Mahalingam and others.

Question 1213.

(V. RAMASWAMI AIYAR) :—Prove that the pedal circle of any point P with respect to a triangle ABC cuts the side BC at an angle equal to the complement of the sum of the angles PBA, PAB and PCB.

Additional solution by K. Srinivasa Raghavan, S. Audinarayanan, N. Sundaram Aiyar and V. V. Satyanarayana Murthi.

Let TXT′ be the tangent to the pedal circle at X.

To prove that **∠BXT is the complement of ∠PAB+∠PBA+∠PCB.**

Now, BZPX, AZPY, CXPY are three cyclic quadrilaterals.

$\therefore \angle PAB = \angle ZYP$; $\angle PBA = \angle PXZ$; $\angle PCB = \angle PYX$.

$$\therefore \angle PAB + \angle PBA + \angle PCB = \angle ZYP + \angle PYX + \angle PXZ$$
$$= \angle ZYX + \angle PXZ.$$

But $\angle ZXT = \angle ZYX$ in the alternate segment.

Hence $P\hat{A}B + P\hat{B}A + P\hat{C}B = P\hat{X}Z + Z\hat{X}T = P\hat{X}T$ which is the complement of $B\hat{X}T$, since $\angle BXP$ is equal to one right angle.

Question 1217.

(K. J. SANJANA) :—(*a*) In a plane triangle ABC, if w is the Brocard angle, prove that

$$\sin^2 w \{ \cot^3 w + \cot (A - w) \cot (B - w) \cot (C - w) \}$$
$$= \operatorname{cosec} A \sin B \sin C + \operatorname{cosec} B \sin C \sin A + \operatorname{cosec} C \sin A \sin B.$$

(*b*) If G is the centroid, prove that

$$\cot GAB \cot GBC \cot GCA + \cot GAC \cot GCB \cot GBA = 6 \cot w.$$

Solution by Martyn M. Thomas, N. Sundaram Aiyar, and S. Audinarayanan.

(*a*) Formulæ used :

$$\cot w = \cot A + \cot B + \cot C \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$\operatorname{cosec}^2 w = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C \quad \ldots \quad \ldots \quad (2)$$

$$\Sigma \cot A \cot B = 1. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Left side expression

$$= \sin w \left\{ \cot^8 w + \frac{1 + \cot A \cot w}{\cot w - \cot A} \cdot \frac{1 + \cot B \cot w}{\cot w - \cot B} \cdot \frac{1 + \cot C \cot w}{\cot w - \cot C} \right\}$$

$$= \sin^2 w \left\{ \cot^2 w + \frac{1 + \cot w (\cot w) + \cot^2 w (1) + \cot^2 w \cot A \cot B \cot C}{\cot^3 w + \cot^2 w (-\cot w) + \cot w (1) - \cot A \cot B \cot C} \right\}$$

$$= \sin^2 w \left\{ \cot^2 w + \frac{1 + 2\cot^2 w + \cot^3 w \cot A \cot B \cot C}{\cot w - \cot A \cot B \cot C} \right\}$$

$$= \sin^2 w \left\{ \frac{1 + 2\cot^2 w + \cot^4 w}{\cot w - \cot A \cot B \cot C} \right\}$$

$$= \frac{\sin^2 w (1 + \cot^2 w)^2}{\text{denominator of the first step}}$$

$$= \frac{\operatorname{cosec}^3 w}{(\cot w - \cot A)\ (\cot w - \cot B)\ (\cot w - \cot C)}$$

$$= \frac{\operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C}{(\cot B + \cot C)\ (\cot C + \cot A)\ (\cot A + \cot B)}$$

$$= \frac{\operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C}{\dfrac{\sin (B + C) \cdot \sin (C + A) \cdot \sin (A + B)}{\sin^2 A \sin^2 B \sin^2 C}}$$

$$= (\operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C) \sin A \sin B \sin C$$

$$= \operatorname{cosec} A \sin B \sin C + \operatorname{cosec} B \sin C \sin A + \operatorname{cosec} C \sin A \sin B.$$

(*b*) Let the median AD make angles α, α' with AB, AC.

Then $1 = \dfrac{BD}{DC} = \dfrac{BD}{DA} \cdot \dfrac{DA}{DC} = \dfrac{\sin \alpha}{\sin B} \cdot \dfrac{\sin C}{\sin (A - \alpha)}$.

$$\therefore \frac{\sin (A - \alpha)}{\sin \alpha} = \frac{\sin C}{\sin B}.$$

$$\therefore \sin A \cot \alpha - \cos A = \frac{\sin C}{\sin B}.$$

$$\therefore \cot \alpha = \cot A + \frac{\sin C}{\sin B \sin A}$$

$$= \cot A + \frac{\sin (A + B)}{\sin B \sin A} = 2 \cot A + \cot B.$$

Similarly, $\cot \alpha' = 2 \cot A + \cot C$.

$\therefore \quad \cot \alpha \cot \beta \cot \gamma + \cot \alpha' \cot \beta' \cot \gamma'$.

$$= (2 \cot A + \cot B)(2 \cot B + \cot C)(2 \cot C + \cot A)$$

$$+ (2 \cot A + \cot C)(2 \cot B + \cot A) \times (2 \cot C + \cot B).$$

$$= (\cot w + \cot A - \cot C)(\cot w + \cot B - \cot A)$$

$$\times (\cot w + \cot C - \cot B) + (\cot w + \cot A - \cot B)$$

$$\times (\cot w + \cot B - \cot C)(\cot w + \cot C - \cot A)$$

$$= \cot^3 w + \cot^2 w\,(0) + \cot w \{ \Sigma(\cot A - \cot C)(\cot B - \cot A) \}$$

$$+ (\cot A - \cot C)(\cot B - \cot A)(\cot C - \cot B)$$

$$+ \cot^3 w + \cot^2 w\,(0) + \cot w \{ \Sigma(\cot C - \cot A)(\cot A - \cot B) \}$$

$$+ (\cot C - \cot A)(\cot A - \cot B)(\cot B - \cot C)$$

$$= 2 \cot^3 w + 2 \cot w \{ \Sigma \cot A \cot B - \Sigma \cot^2 A \}$$

$$= 2 \cot^3 w + 2 \cot w \{ 1 - (\Sigma \cot A)^2 + 2 \Sigma \cot A \cot B \}$$

$$= 2 \cot^3 w + 2 \cot w \{ 1 - \cot^2 w + 2) \}$$

$$= 6 \cot w.$$

Questions for Solution.

1280. (M. BHEEMASENA RAO AND THE LATE C. KRISHNAMACHARI) :—

If $S_r = 1 + \frac{1}{2^r} + \frac{1}{3^r} + \ldots\ldots$ and γ is Euler's constant, show that

$$\int_0^\infty e^{-x} (\log x)^3 \, dx = -(\gamma^3 + 3\gamma S_2 + 2S_3);$$

$$\int_0^\infty e^{-x} (\log x)^4 dx = \gamma^4 + 6\gamma^2 S_2 + 8\gamma S_3 + 3S_2^2 + 6S_4;$$

and show how to evaluate generally

$$\int_0^\infty e^{-x} \; (\log x)^n \, dx.$$

1281. (M. BHEEMASENA RAO AND THE LATE C. KRISHNAMACHARI):—

If $S_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)\frac{1}{n}$, show that

$$\lim_{n=\infty} \left\{ (S_1 + S_2 + \ldots + S_n) - \gamma \log n - \tfrac{1}{2} (\log n)^2 \right\} = \frac{\gamma^2}{2} + \frac{\pi^2}{12}.$$

Further show that

$$S_1 + S_2 + \ldots + S_n = \binom{n}{1} - \binom{n}{2}\frac{1}{2^2} + \binom{n}{3}\frac{1}{3^2} - \ldots + (-1)^{n-1} \binom{n}{n}\frac{1}{n^2},$$

where $\binom{n}{r}$ = number of combinations of n things r at a time.

1282. (M. BHEEMASENA RAO AND THE LATE C. KRISHNAMACHARI) :—

If $A_r = \dfrac{2r\ C_r}{(r+1)}$, show that

$$[A_r, A_{n+r-1}] = 1!\,3!\,5!\ldots(2n-1)!$$

$$\times \frac{(2r)!\,(2r+2)!\ldots(2r+2n-2)!}{r!\,(r+1)!\ldots(r+2n-1)!}$$

$$= \left(\frac{m+1}{1}\right)^1 \left(\frac{m+2}{2}\right)^2 \left(\frac{m+r-2}{r-2}\right)^2 \left(\frac{m+r-1}{r-1}\right)^1$$

$$\times \left(\frac{2m+3}{3}\right)^1 \left(\frac{2m+5}{5}\right)^2 \left(\frac{2m+2r-5}{2r-5}\right)^2$$

$$\times \left(\frac{2m+2r-3}{2r-3}\right)^1,$$

where the indices increase in order and then diminish.

1283. (K. J. SANJANA, M.A.) :—A conic is circumscribed to the triangle of reference.

(i) If its centre be at the point (fgh), prove that its equation is

$$\Sigma f\,(bg + ch - af) \div \alpha = 0;$$

(ii) If one focus be at (fgh), the equation is

$$\Sigma f\ (mg + nh - lf) \div \alpha = 0,$$

where l, m, n are the sides of the pedal triangle of the given focus.

[The first part was published in the Mathematical columns of the *Educational Times* for Nov. 1920.]

1284. (G. V. TELANG) :—Show that the common chord of the pedal circle of a Brocard point and the N. P. circle subtends at the N.P. centre an angle equal to 4ω

Also, show that any triangle and that formed by its three Brocard points have the same centroid.

195324

LIST OF JOURNALS RECEIVED BY THE SOCIETY.

1 Messenger of Mathematics
2 Quarterly Journal of Mathematics
3 Mathematical Gazette
4 The Annals of Mathematics
5 American Journal of Mathematics
6 Bulletin of the American Mathematical Society
7 Transactions of the American Mathematical Society
8 Monthly Notices of the Royal Astronomical Society
9 Proceedings of the Royal Society of London
10 The Philosophical Magazine and Journal of Science
11 Astrophysical Journal
12 Crelle's Journal
13 L'intermediaere des Mathematiciens
14 Mathematische Annalen
15 Philosophical Transactions of the Royal Society of London
16 Acta Mathematica
17 Popular Astronomy
18 Proceedings of the Edinburgh Mathematical Society
19 Proceedings of the London Mathematical Society
20 Mathematics Teacher
21 Bulletin of the Calcutta Mathematical Society
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24 The American Mathematical Monthly
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